ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT

Journal of Algebra 267 (2003) 377–395

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

On graded Frobenius algebras

Takayoshi Wakamatsu

Faculty of Education, Saitama University, Shimo-Ohkubo 255, Saitama 338-8570, Japan

Received 14 November 1997

Communicated by Kent R. Fuller

1. Introduction

Frobenius algebras appear often in mathematics, for example, group algebras of finite groups and restricted enveloping algebras of restricted Lie algebras, more generally, finite-dimensional Hopf algebras are all Frobenius. Recently, relating to Artin–Shelter's regular algebras or quantized polynomial rings, Frobenius algebras appear quite naturally. For details, see Artin and Schelter [1] and Smith [4]. Hence, it is important to know the way of obtaining Frobenius algebras. In the previous paper [5], over an algebraically closed field, the author proved that any Frobenius algebra without semisimple part can be constructed from a nilpotent Frobenius algebra. However, it is not so easy to construct possible nilpotent Frobenius algebras generally. So, in this paper, we concentrate on graded Frobenius algebras and construct them as factors of tensor algebras.

Let A be an algebra over an algebraically closed field K and ${}_AX_A$ a bimodule. For an automorphism $\sigma \in \mathrm{Aut}_K(A)$ and an isomorphism $\gamma : {}_AX_A \xrightarrow{\sim} {}_\sigma X_\sigma$, we consider a linear map $\theta : X^{\otimes d} \to K$, where ${}_\sigma X_\sigma$ stands for the twisted bimodule of ${}_AX_A$ by σ and $\otimes$ means $\otimes_A$. We call the triple $\Phi = (\sigma, \gamma, \theta)$ an admissible system for a graded Frobenius algebra if θ satisfies

$$\theta(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = \theta\big(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1)\big)$$

for any element $x_1 \otimes x_2 \otimes \cdots \otimes x_d \in X^{\otimes d}$. In Section 2, from an admissible system $\Phi = (\sigma, \gamma, \theta)$, we derive a graded Frobenius algebra $\Lambda(\Phi) = \bigoplus_{i=0}^{d} \Lambda_i$ with two properties: (H1) $\Lambda_i \cdot \Lambda_j = \Lambda_{i+j}$ for any i, j, and (H2) $\Lambda_d \supseteq \mathrm{soc}(\Lambda)$, which is a factor of the tensor algebra $T_A(X) = A \oplus X \oplus X^{\otimes 2} \oplus \cdots$ by a homogeneous ideal. Conversely, it is proved that any basic graded Frobenius algebra Λ possessing above two properties is

E-mail address: wakataka@d4.dion.ne.jp.

doi:10.1016/S0021-8693(03)00262-X

isomorphic to $\Lambda(\Phi)$ for some admissible system $\Phi = (\sigma, \gamma, \theta)$, in Section 3. Martinéz-Villa [3] proved that a basic graded Frobenius algebra $A = \bigoplus_{i=1}^{d} \Lambda_i$ with the properties $\operatorname{rad} \Lambda = \bigoplus_{i=1}^{d} \Lambda_i$ and $\Lambda_1 \cong \operatorname{rad} \Lambda / \operatorname{rad}^2 \Lambda$ satisfies $\Lambda_d = \operatorname{soc}(\Lambda)$. Therefore, many graded Frobenius algebras including radical graded algebras are of the form $\Lambda(\Phi)$ for admissible systems $\Phi = (\sigma, \gamma, \theta)$.

In Section 4, we study the case $A = K$. In this case, admissible system $\Phi = (\sigma, \gamma, \theta)$ is quite simple because σ is 1_K the identity map of K and γ just a K-automorphism of a vector space X, i.e., $\gamma \in GL(X)$. We call the map θ nondegenerate if $\operatorname{Ker}(T_K(X) \to \Lambda(\Phi)) \subseteq (X^{\otimes 2}) = X^{\otimes 2} \oplus X^{\otimes 3} \oplus \cdots$. For two admissible systems $\Phi = (1_K, \gamma, \theta)$ and $\Phi' = (1_K, \gamma', \theta')$ with θ and θ' nondegenerate, it is proved that the algebras $\Lambda(\Phi)$ and $\Lambda(\Phi')$ are isomorphic over K if and only if $\gamma' = s^{-1} \circ \gamma \circ s$ and $\theta' = \theta \circ s^{\otimes d}$ for some element $s \in GL(X)$. Let us put $H \subseteq \operatorname{Hom}_K(X^{\otimes d}, K)$ to be the subset consisting of all linear maps $\theta : X^{\otimes d} \to K$ such that $(1_K, \gamma, \theta)$ is an admissible system for some $\gamma \in GL(X)$. It is checked that $GL(X)$ acts on H by defining $\theta \overset{s}{\mapsto} \theta \circ s^{\otimes d}$ for $\theta \in H$ and $s \in GL(X)$. Then, we know that classifying graded Frobenius algebras Λ with the properties $\Lambda / \operatorname{rad} \Lambda \cong K$, $\operatorname{rad} \Lambda / \operatorname{rad}^2 \Lambda \cong X$, and $\operatorname{rad}^d \Lambda = 0$ is equivalent to classifying $GL(X)$-orbits of nondegenerate points θ in the set H. Further, in Section 5, we consider some concrete constructions of admissible systems $(1_K, \gamma, \theta)$ over K.

Troughout this paper, except the tensor algebra $T_A(X)$, all algebras and modules are finite-dimensional over the ground field K.

2. Admissible system

Let A be a finite-dimensional algebra over a field K and $\sigma \in \operatorname{Aut}_K(A)$ a K-algebra automorphism. For a module M_A, we denote by M_σ the σ-twisted module. Thus, M_σ is the same with M_A as a K-space and the action of A to M_σ is given by $M_\sigma \times A \ni (m, a) \mapsto m\sigma(a) \in M_\sigma$. Similarly, the module ${}_\sigma N$ is defined for a left module ${}_A N$. We denote the ordinary duality functor $\operatorname{Hom}_K(?, K)$ by D. Now assume that we are given a bimodule isomorphism

$$\gamma : {}_A X_A \xrightarrow{\sim} {}_\sigma X_\sigma$$

and a linear map

$$\theta : X^{\otimes d} \to K,$$

where $d \geqslant 1$ and $\otimes$ means $\otimes_A$, the tensor product over A.

Definition 2.1. We call a triple $\Phi = (\sigma, \gamma, \theta)$ an admissible system for a graded Frobenius algebra if the condition

$$\theta(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = \theta\big(x_2 \otimes \cdots x_d \otimes \gamma(x_1)\big)$$

is satisfied for any element $x_1 \otimes x_2 \otimes \cdots \otimes x_d \in X^{\otimes d}$.

Remark 2.2. A linear map $\theta : X^{\otimes d} \to K$ with the property

$$\theta(ax_1 \otimes \cdots \otimes x_d) = \theta\big(x_1 \otimes \cdots \otimes x_d \sigma(a)\big)$$

for all $a \in A$ and $x_1 \otimes \cdots \otimes x_d \in X^{\otimes d}$ determines a bimodule homomorphism $\varphi : {}_A(X^{\otimes d})_A \to {}_\sigma D(A)_A$ by

$$\varphi(x_1 \otimes \cdots \otimes x_d)(a) = \theta(x_1 \otimes \cdots \otimes x_d a)$$

for $a \in A$ and $x_1 \otimes \cdots \otimes x_d \in X^{\otimes d}$. Conversely, a bimodule homomorphism φ determines a linear map θ satisfying the above condition by

$$\theta(x_1 \otimes \cdots \otimes x_d) = \varphi(x_1 \otimes \cdots \otimes x_d)(1).$$

Then, by using φ, the defining condition of admissible system is written as

$$\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_d)(1) = \varphi\big(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1)\big)(1).$$

We consider sometimes an admissible system as a triple $\Phi = (\sigma, \gamma, \varphi)$, using φ instead of θ.

In what follows, we put ${}_A(X^{\otimes 0})_A = {}_A A_A$ for convenience. For each integer $0 \leqslant i \leqslant d$, the bimodule homomorphism

$$\varphi_i : {}_A\big(X^{\otimes i}\big)_A \to {}_\sigma \operatorname{Hom}_{\text{mod-}A}\big({}_A\big(X^{\otimes (d-i)}\big)_A, {}_\sigma D(A)_A\big)_A$$

is defined by $\varphi_i(y)(z) = \varphi(y \otimes z)$. Similarly, the bimodule homomorphism

$$\theta_i : {}_A\big(X^{\otimes i}\big)_A \to {}_\sigma D\big(X^{\otimes (d-i)}\big)_A$$

is defined by $\theta_i(y)(z) = \theta(y \otimes z) = \varphi(y \otimes z)(1)$. The following is proved easily.

Lemma 2.3. $\operatorname{Ker}(\varphi_i) = \operatorname{Ker}(\theta_i)$.

The homomorphisms

$$\varphi_i^* : {}_A\big(X^{\otimes i}\big)_A \to {}_\sigma \operatorname{Hom}_{\text{mod-}A}\big({}_A\big(X^{\otimes (d-i)}\big)_A, {}_A D(A)_{\sigma^{-1}}\big)_{\sigma^{-1}}$$

and

$$\theta_i : {}_A\big(X^{\otimes i}\big)_A \to {}_A D\big(X^{\otimes (d-i)}\big)_{\sigma^{-1}}$$

are defined by $\varphi_i^*(y)(z) = \varphi(z \otimes y)$ and $\theta_i^*(y)(z) = \theta(z \otimes y) = \varphi(z \otimes y)(1)$, respectively. It should be noted that $\varphi_i(y)(z) = \varphi_{d-i}^*(z)(y)$ and $\theta_i(y)(z) = \theta_{d-i}^*(z)(y)$ hold.

Lemma 2.4. *The following hold*:

(1) $\mathrm{Ker}(\varphi_i) = \mathrm{Ker}(\varphi_i^*)$, *and*
(2) $\sigma(\mathrm{Ker}(\varphi_0)) = \mathrm{Ker}(\varphi_0)$ *and* $\gamma^{\otimes i}(\mathrm{Ker}(\varphi_i)) = \mathrm{Ker}(\varphi_i)$ *for all* $i > 0$.

Proof. From the equality

$$\begin{aligned}\varphi\big(aX^{\otimes d}\big)(1) &= \big(\sigma(a)\varphi\big(X^{\otimes d}\big)\big)(1) = \varphi\big(X^{\otimes d}\big)\big(\sigma(a)\big) = \big(\varphi\big(X^{\otimes d}\big)\sigma(a)\big)(1)\\ &= \varphi\big(X^{\otimes d}\sigma(a)\big)(1),\end{aligned}$$

we get $\sigma(\mathrm{Ker}(\varphi_0)) = \mathrm{Ker}(\varphi_0^*)$. Similarly, from

$$\begin{aligned}\varphi\big(aX^{\otimes d}\big)(1) &= \varphi\big(X^{\otimes(d-1)} \otimes \gamma(aX)\big)(1)\\ &= \varphi\big(X^{\otimes(d-2)} \otimes \gamma(aX) \otimes \gamma(X)\big)(1)\\ &\;\;\vdots\\ &= \varphi\big(\gamma(aX) \otimes \gamma(X)^{\otimes(d-1)}\big)(1)\\ &= \varphi\big(\sigma(a)X^{\otimes d}\big)(1),\end{aligned}$$

we obtain $\sigma(\mathrm{Ker}(\varphi_0)) = \mathrm{Ker}(\varphi_0)$ and, therefore, $\mathrm{Ker}(\varphi_0) = \mathrm{Ker}(\varphi_0^*)$. For each $i \geqslant 1$ and $y \in X^{\otimes i}$, from

$$\begin{aligned}\varphi\big(y \otimes X^{\otimes(d-i)}\big)(1) &= \varphi\big(X^{\otimes(d-i)} \otimes \gamma^{\otimes i}(y)\big)(1) = \varphi\big(\gamma^{\otimes i}(y) \otimes \gamma^{\otimes(d-i)}\big(X^{\otimes(d-i)}\big)\big)(1)\\ &= \varphi\big(\gamma^{\otimes i}(y) \otimes X^{\otimes(d-i)}\big)(1),\end{aligned}$$

we have $\gamma^{\otimes i}(\mathrm{Ker}(\varphi_i)) = \mathrm{Ker}(\varphi_i^*) = \mathrm{Ker}(\varphi_i)$. □

Let us put $R_i = \mathrm{Ker}(\varphi_i) = \mathrm{Ker}(\varphi_i^*)$ for each $0 \leqslant i \leqslant d$. By the above lemma, in the tensor algebra

$$T_A(X) = A \oplus X \oplus X^{\otimes 2} \oplus \cdots,$$

we have a two-sided ideal

$$R(\Phi) = R_0 \oplus R_1 \oplus \cdots \oplus R_d \oplus X^{\otimes(d+1)} \oplus X^{\otimes(d+2)} \oplus \cdots,$$

since the relations $\mathrm{Ker}(\varphi_i) \otimes X \subseteq \mathrm{Ker}(\varphi_{i+1})$ and $X \otimes \mathrm{Ker}(\varphi_i^*) \subseteq \mathrm{Ker}(\varphi_{i+1}^*)$ hold.

Definition 2.5. We denote by $\Lambda(\Phi)$ the factor algebra

$$T_A(X)/R(\Phi) = A/R_0 \oplus X/R_1 \oplus \cdots \oplus X^{\otimes d}/R_d.$$

We put $\overline{A} = A/R_0$ and $\overline{X} = X/R_1$. Then, by Lemma 2.4, σ and γ induce an automorphism $\bar{\sigma} \in \mathrm{Aut}_K(\overline{A})$ and a bimodule isomorphism $\bar{\gamma} : {}_{\overline{A}}\overline{X}_{\overline{A}} \xrightarrow{\sim} {}_{\bar{\sigma}}\overline{X}_{\bar{\sigma}}$, respectively. Since $\varphi(X^{\otimes d})(R_0) = \varphi(R_0 X^{\otimes d})(1) = \varphi_0(R_0)(X^{\otimes d}) = 0$, we have a commutative diagram

$$\begin{array}{ccc} {}_A\big(X^{\otimes d}\big)_A & \xrightarrow{\varphi} & {}_\sigma D(A)_A \\ {\scriptstyle \alpha^{\otimes d}}\big\downarrow & & \big\uparrow{\scriptstyle D(\beta)} \\ {}_{\overline{A}}\big(\overline{X}^{\otimes d}\big)_{\overline{A}} & \xrightarrow{\bar{\varphi}} & {}_{\bar{\sigma}}D(\overline{A})_{\overline{A}}, \end{array}$$

where $\alpha : X \to \overline{X}$ and $\beta : A \to \overline{A}$ are the projection maps. It is not hard to see that the morphism $\bar{\varphi}$ is surjective and the triple $\overline{\Phi} = (\bar{\sigma}, \bar{\gamma}, \bar{\varphi})$ becomes an admissible system over $\overline{A}$ and $\overline{X}$. We call the new system $\overline{\Phi}$ the reduced system of Φ. Further, for each $0 \leqslant i \leqslant d$, we have the commutative diagram

$$\begin{array}{ccc} X^{\otimes i} & \xrightarrow{\varphi_i} & \mathrm{Hom}_A\big(X^{\otimes(d-i)}, D(A)\big) \\ {\scriptstyle \alpha^{\otimes i}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{Hom}(\alpha^{\otimes(d-i)}, D(\beta))} \\ \overline{X}^{\otimes i} & \xrightarrow[\bar{\varphi}_i]{} & \mathrm{Hom}_{\overline{A}}\big(\overline{X}^{\otimes(d-i)}, D(\overline{A})\big), \end{array}$$

where the map $\alpha^{\otimes i}$ is surjective and $\mathrm{Hom}(\alpha^{\otimes(d-i)}, D(\beta))$ is injective. Therefore, we have an isomorphism

$$X^{\otimes i}/R_i \cong \mathrm{Im}(\varphi_i) \cong \mathrm{Im}(\bar{\varphi}_i) \cong \overline{X}^{\otimes i}/\overline{R}_i$$

for each $0 \leqslant i \leqslant d$. Hence, we get the following result.

Proposition 2.6. $\Lambda(\overline{\Phi}) \cong \Lambda(\Phi)$.

For an admissible system $\Phi = (\sigma, \gamma, \varphi)$, we call the homomorphism φ or the system Φ nondegenerate if $R_1 = 0$ holds. It is easy to see that the following assertions hold:

(1) $R_0 = 0$ if and only if $\mathrm{Im}(\varphi) = D(A)$;
(2) $R_0 = 0 = R_1$ if and only if $\overline{\Phi} = \Phi$.

In the second case, we call the system Φ reduced. Therefore, nondegenerate system Φ with the surjective homomorphism φ is reduced.

Theorem 2.7. *The graded algebra $\Lambda(\Phi)$ is Frobenius.*

Proof. For each $0 \leqslant i \leqslant d$, it is checked that the map θ_i induces an injective homomorphism

$$\bar{\theta}_i : X^{\otimes i}/R_i \to D\big(X^{\otimes (d-i)}/R_{d-i}\big).$$

Therefore, the linear map

$$\bar{\theta} = \begin{bmatrix} \bar{\theta}_0 & & & & \mathbf{0} \\ & \bar{\theta}_1 & & & \\ & & \ddots & & \\ & & & \bar{\theta}_{d-1} & \\ \mathbf{0} & & & & \bar{\theta}_d \end{bmatrix} : \Lambda(\Phi) \to D\big(\Lambda(\Phi)\big)$$

is bijective. On the other hand, the element $\bar{\theta}_0(1) \in D(X^{\otimes d}/R_d) \subseteq D(\Lambda(\Phi))$ defines the right $\Lambda(\Phi)$-module homomorphism

$$\big(\bar{\theta}_0(1)?\big) : \Lambda(\Phi)_{\Lambda(\Phi)} \to D\big(\Lambda(\Phi)\big)_{\Lambda(\Phi)},$$

and easy calculation shows that the map $(\bar{\theta}_0(1)?)$ coincides with $\bar{\theta}$. $\square$

By an easy calculation, the Nakayama automorphism $\nu \in \mathrm{Aut}_K(\Lambda(\Phi))$ corresponding to the above isomorphism is given as follows.

Proposition 2.8. *The Nakayama automorphism ν sends an element*

$$(\lambda_0, \lambda_1, \ldots, \lambda_d) \in A/R_0 \oplus X/R_1 \oplus \cdots \oplus X^{\otimes d}/R_d = \Lambda(\Phi)$$

to $(\bar{\sigma}(\lambda_0), \bar{\gamma}(\lambda_1), \ldots, \bar{\gamma}^{\otimes d}(\lambda_d)) \in \Lambda(\Phi)$.

Remark 2.9. In the case $d = 0$, if we understand that the map φ is from ${}_A A_A$ to ${}_\sigma D(A)_A$ and that $\Lambda(\Phi) = \overline{A}$, this is Frobenius also. Moreover, for $d = 1$, the algebra $\Lambda(\Phi)$ is just the trivial extension of $\overline{A}$ by the bimodule ${}_{\bar{\sigma}} D(\overline{A})_{\overline{A}}$. So, our construction stated in this section is a generalization of that of trivial extension self-injective algebras.

Remark 2.10. For an admissible system $\Phi = (\sigma, \gamma, \varphi)$, even if $R_0 \neq 0$, we can define Frobenius algebra

$$\Lambda'(\Phi) = A \oplus X/R_1 \oplus \cdots \oplus X^{\otimes (d-1)}/R_{d-1} \oplus {}_\sigma D(A)_A$$

in the same way as $\Lambda(\Phi)$. In this algebra, Nakayama automorphism ν' sends an element

$$(a, \lambda_1, \ldots, \lambda_{d-1}, f) \in A \oplus X/R_1 \oplus \cdots \oplus X^{\otimes (d-1)}/R_{d-1} \oplus D(A)$$

to $(\sigma(a), \bar{\gamma}(\lambda_1), \ldots, \bar{\gamma}^{\otimes (d-1)}(\lambda_{d-1}), f \circ \sigma^{-1})$.

3. Structure of graded Frobenius algebras

In this section, we prove that any graded Frobenius algebra satisfying the conditions below is constructed from a suitable admissible system. Let K be algebraically closed and Λ a basic Frobenius K-algebra. We put for Λ the assumptions

(H1) $\Lambda = \bigoplus_{i=0}^{d} \Lambda_i$ as a K-space and $\Lambda_i \cdot \Lambda_j = \Lambda_{i+j}$ for each i, j, and
(H2) $\Lambda_d \supseteq \operatorname{soc}(\Lambda)$.

As quoted in introduction, Martinéz-Villa [3] proved that a basic graded Frobenius algebra $\Lambda = \bigoplus_{i=0}^{d} \Lambda_i$ satisfying $\operatorname{rad} \Lambda = \bigoplus_{i \geqslant 1} \Lambda_i$ and $\operatorname{rad}^2 \Lambda = \bigoplus_{i \geqslant 2} \Lambda_i$ has the property $\operatorname{soc}(\Lambda) = \Lambda_d$. Hence, the assumption (H2) is natural.

Let $1 = e_1 + e_2 + \cdots + e_n$ be a decomposition of the identity $1 \in \Lambda$ into the orthogonal primitive idempotents. Then, $\operatorname{soc}(e_i \Lambda) \cong K$ for each i and, from $\operatorname{soc}(\Lambda) = \bigoplus_{i=1}^{n} \operatorname{soc}(e_i \Lambda) \cong \bigoplus^{n} K$, we have a linear map $q' : (a_i)_i \mapsto \sum_{i=1}^{n} a_i$. By giving a K-space complement of $\operatorname{soc}(\Lambda)$ in Λ_d, we have a linear map $q \in D(\Lambda_d)$ which is the same with q' on $\operatorname{soc}(\Lambda)$ and zero on the complement. Since we have a K-space decomposition of Λ in the assumption (H1), we may consider $q \in D(\Lambda_d) \subseteq D(\Lambda)$.

Lemma 3.1. *The homomorphism $(q?) : \Lambda_\Lambda \to D(\Lambda)_\Lambda$ is bijective.*

Proof. We have to prove that $q\lambda = 0$ implies $\lambda = 0$ for $\lambda \in \Lambda$. It is checked that the following implications hold:

$$q\lambda = 0 \quad \Leftrightarrow \quad q(\lambda\Lambda) = 0 \quad \Rightarrow \quad \lambda\Lambda \cap \operatorname{soc}(\Lambda) = 0 \quad \Leftrightarrow \quad \lambda = 0. \qquad \square$$

Lemma 3.2. *Since the relation $q\Lambda_i \subseteq D(\Lambda_{d-i})$ holds, $(q?)$ gives a bijective map $\Lambda_i \to D(\Lambda_{d-i})$ for each i.*

We denote by $\nu \in \operatorname{Aut}_K(\Lambda)$ the Nakayama automorphism corresponding to the isomorphism $(q?) : \Lambda_\Lambda \to D(\Lambda)_\Lambda$.

Lemma 3.3. *$\nu(\Lambda_i) = \Lambda_i$ for each i.*

Proof. By the definition of Nakayama automorphism, we have the following commutative diagram for each element $\lambda \in \Lambda$:

$$\begin{array}{ccc} \Lambda_\Lambda & \xrightarrow{(q?)} & D(\Lambda)_\Lambda \\ {\scriptstyle (\lambda?)}\downarrow & & \downarrow{\scriptstyle (\nu(\lambda)?)} \\ \Lambda_\Lambda & \xrightarrow[(q?)]{} & D(\Lambda)_\Lambda. \end{array}$$

For an element $\lambda \in \Lambda_i$, we have $\lambda\Lambda_j \subseteq \Lambda_{i+j}$ and $q\lambda\Lambda_j \subseteq q\Lambda_{i+j} = D(\Lambda_{d-i-j})$. Therefore, $(\nu(\lambda)?)$ sends the space $D(\Lambda_{d-j})$ into $D(\Lambda_{d-i-j})$ and we see $\nu(\lambda) \in \Lambda_i$. $\square$

Now we have an algebra Λ_0, an automorphism $\sigma = \nu|_{\Lambda_0} \in \mathrm{Aut}_K(\Lambda_0)$, a bimodule isomorphism $\gamma = \nu|_{\Lambda_1} : {}_{\Lambda_0}(\Lambda_1)_{\Lambda_0} \xrightarrow{\sim} {}_\sigma D(\Lambda_1)_{\Lambda_0}$ and a surjective bimodule homomorphism

$$\varphi : {}_{\Lambda_0}\big(\Lambda_1^{\otimes d}\big)_{\Lambda_0} \xrightarrow{m_d} {}_{\Lambda_0}(\Lambda_d)_{\Lambda_0} \xrightarrow{(q?)} {}_\sigma D(\Lambda_0)_{\Lambda_0},$$

where the surjection m_d is given by the multiplication of Λ and $\otimes$ means $\otimes_{\Lambda_0}$.

Proposition 3.4. *The triple $(\sigma, \gamma, \varphi)$ defined above is a reduced admissible system over Λ_0 and Λ_1.*

Proof. We only have to check the defining condition of φ. We have

$$\begin{aligned}\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_d)(1) &= q(x_1x_2\cdots x_d) = (qx_1)(x_2\cdots x_d) = \big(\nu(x_1)q\big)(x_2\cdots x_d)\\ &= q\big(x_2\cdots x_d\nu(x_1)\big) = \varphi\big(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1)\big)(1). \qquad \square\end{aligned}$$

Theorem 3.5. *$\Lambda \cong \Lambda(\sigma, \gamma, \varphi)$ as K-algebras.*

Proof. Denote by m_i the surjective map $\Lambda_1^{\otimes i} \to \Lambda_i$, denned by the multiplication of Λ. For any elements $y \in \Lambda_1^{\otimes i}$ and $z \in \Lambda_1^{\otimes (d-i)}$, we have

$$\varphi_i(y)(z) = qm_i(y)m_{d-i}(z) \in D(\Lambda_0),$$

and get $\mathrm{Ker}(\varphi_i) = \mathrm{Ker}(m_i)$. Hence, the map m_i induces the isomorphism $\bar{m}_i : \Lambda_1^{\otimes i}/\mathrm{Ker}(\varphi_i) \xrightarrow{\sim} \Lambda_i$ for each i. Then, the map

$$\begin{bmatrix} \bar{m}_0 & & & & \mathbf{0} \\ & \bar{m}_1 & & & \\ & & \ddots & & \\ & & & \bar{m}_{d-1} & \\ \mathbf{0} & & & & \bar{m}_d \end{bmatrix} : \Lambda(\sigma, \gamma, \varphi) \to \Lambda$$

becomes a K-algebra isomorphism. $\square$

4. Admissible system over *K*

In this section, we study admissible systems $\Phi = (\sigma, \gamma, \theta)$ over $A = K$ and ${}_AX_A$. In this case, $\sigma = 1_K$ and X is just a K-space. Therefore, $\gamma \in GL(X)$ and $\theta : X^{\otimes d} \to K$ is a linear map satisfying $\theta(x_1 \otimes x_2 \otimes \cdots \otimes x_d) = \theta(x_2 \otimes \cdots \otimes x_d \otimes \gamma(x_1))$.

Let $e_1, e_2, \ldots, e_n$ be a K-basis of X and $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ the dual basis of $D(X)$. Then, $\theta \in D(X^{\otimes d}) \cong D(X)^{\otimes d}$ can be written as

$$\theta = \sum_{i_1, i_2, \ldots, i_d} \Theta(i_1, i_2, \ldots, i_d)\hat{e}_{i_1} \otimes \hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_d}$$

with $\Theta(i_1, i_2, \ldots, i_d) = \theta(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}) \in K$. We define the coefficients $\Gamma(i, j) \in K$ by

$$\gamma(e_j) = \sum_{i=1}^{n} e_i \Gamma(i, j).$$

Then, the condition of θ is written as

$$\Theta(i_1, i_2, \ldots, i_d) = \sum_{i=1}^{n} \Theta(i_2, \ldots, i_d, i)\Gamma(i, i_1) \tag{$\star$}$$

for any $1 \leqslant i_1, i_2, \ldots, i_d \leqslant n$.

The coefficients $\Gamma(i, j)$ determine the $n \times n$ matrix Γ with the (i, j)-entry $\Gamma(i, j)$. Similarly, we may consider that the set of coefficients $\{\Theta(i_1, i_2, \ldots, i_d)\}$ determines a generalized matrix Θ. So, giving a generalized matrix Θ is equivalent to giving a linear map $\theta \in D(X)^{\otimes d} \cong D(X^{\otimes d})$ and the above equality $(\star)$ is the condition for θ to be a map for which the triple $\sigma = (1_K, \gamma, \theta)$ is an admissible system. We define the product $\Theta \cdot \Gamma$ of a generalized matrix Θ with a usual matrix Γ by

$$(\Theta \cdot \Gamma)(i_1, \ldots, i_{d-1}, i_d) = \sum_{i=1}^{n} \Theta(i_1, \ldots, i_{d-1}, i)\Gamma(i, i_d)$$

and the transpose ${}^t\Theta$ of a generalized matrix of Θ by

$${}^t\Theta(i_1, \ldots, i_{d-1}, i_d) = \Theta(i_d, i_1, \ldots, i_{d-1}).$$

Then, the condition $(\star)$ is written simply as ${}^t\Theta = \Theta \cdot \Gamma$.

We next consider, when Θ satisfies the condition $(\star)$, how we can write down the condition for θ to be nondegenerate, in terms of linear algebra. Observe that θ is nondegenerate if and only if the map $\theta_1 : X \to D(X^{\otimes(d-1)})$ is injective. By the equation

$$\theta_1(e_{i_1}) = \sum_{i_2, \ldots, i_d} \Theta(i_1, i_2, \ldots, i_d)\hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_d},$$

we have

$$\theta_1\left(\sum_{i=1}^{n} c_i e_i\right) = \sum_{i_2, \ldots, i_d} \left(\sum_{i=1}^{n} c_i \Theta(i_1, i_2, \ldots, i_d)\right)\hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_d}.$$

Therefore, the map θ is nondegenerate if and only if the following condition is satisfied:

$$\sum_{i-1}^{n} c_i \Theta(i, i_2, \ldots, i_d) = 0 \text{ for all } i_2, \ldots, i_d \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0. \tag{$\star\star$}$$

The following lemma is easily checked.

Lemma 4.1. *Let $(1_K, \gamma, \theta)$ be an admissible system over K and X. Then, for $s \in GL(X)$, the triple $(1_K, s^{-1} \circ \gamma \circ s, \theta \circ s^{\otimes d})$ is also an admissible system. Moreover, if $(1_K, \gamma, \theta)$ is nondegenerate, then so is $(1_K, s^{-1} \circ \gamma \circ s, \theta \circ s^{\otimes d})$.*

Lemma 4.2. *Let $(1_K, \gamma, \theta)$ and $(1_K, \gamma', \theta)$ be admissible systems. If the common map θ is nondegenerate then $\gamma' = \gamma$.*

Proof. The fact that γ and θ form an admissible system $(1_K, \gamma, \theta)$ implies the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\theta_1"] \arrow[d, "\gamma"'] & D\bigl(X^{\otimes(d-1)}\bigr) \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "\theta_1^*"'] & D\bigl(X^{\otimes(d-1)}\bigr). \end{tikzcd}$$

Therefore, we have $\theta_1^* \circ \gamma = \theta_1$. By the same reason, we have $\theta_1^* \circ \gamma' = \theta_1$, and hence, $\theta_1^* \circ \gamma = \theta_1^* \circ \gamma'$. Here, the map θ_1^* is injective since θ is nondegenerate and it follows that $\gamma = \gamma'$. $\square$

Theorem 4.3. *Assume that both triples $\Phi = (1_K, \gamma, \theta)$ and $\Phi' = (1_K, \gamma', \theta')$ are nondegenerate admissible systems. Then, $\Lambda(\Phi)$ and $\Lambda(\Phi')$ are isomorphic as K-algebras if and only if $\theta' = \theta \circ s^{\otimes d}$ (and hence $\gamma' = s^{-1} \circ \gamma \circ s$) for some $s \in GL(X)$.*

Proof. If $\theta' = \theta \circ s^{\otimes d}$, then the map

$$\begin{bmatrix} 1 & & & & \mathbf{0} \\ & s & & & \\ & & \bar{s}^{\otimes 2} & & \\ & & & \ddots & \\ \mathbf{0} & & & & \bar{s}^{\otimes d} \end{bmatrix}$$

is a K-algebra isomorphism from $\Lambda(\Phi')$ to $\Lambda(\Phi)$. Conversely, suppose that we are given a K-algebra isomorphism $\alpha : \Lambda(\Phi') \xrightarrow{\sim} \Lambda(\Phi)$. Write $R_i = \mathrm{Ker}(\theta_i)$ and $R_i' = \mathrm{Ker}(\theta_i')$. Then, according to the K-space decompositions

$$\Lambda(\Phi) = K \oplus X \oplus X^{\otimes 2}/R_2 \oplus \cdots \oplus X^{\otimes d}/R_d$$

and

$$\Lambda\bigl(\Phi'\bigr) = K \oplus X \oplus X^{\otimes 2}/R_2' \oplus \cdots \oplus X^{\otimes d}/R_d',$$

the map α has a matrix presentation

$$\begin{bmatrix} 1 & & & & \mathbf{0} \\ 0 & \alpha_{11} & & & \\ 0 & \alpha_{21} & \alpha_{22} & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & \alpha_{d1} & \alpha_{d2} & \cdots & \alpha_{dd} \end{bmatrix},$$

where $\alpha_{ij} : X^{\otimes j}/R'_j \to X^{\otimes i}/R_i$. The matrix is lower triangular because, for each i, the map α induces a bijection $\mathrm{rad}^i(\Lambda(\Phi')) \xrightarrow{\sim} \mathrm{rad}^i(\Lambda(\Phi))$ and the equality

$$\mathrm{rad}^i\big(\Lambda(\Phi)\big) = X^{\otimes i}/R_i \oplus \cdots \oplus X^{\otimes d}/R_d$$

and

$$\mathrm{rad}^i\big(\Lambda\big(\Phi'\big)\big) = X^{\otimes i}/R'_i \oplus \cdots \oplus X^{\otimes d}/R'_d$$

hold. By the same reason, it is immediate that the map $\alpha_i = \alpha_{ii}$ are all bijective. For any element $x_i \in X^{\otimes i}$, we denote by $\bar{x}_i$ the corresponding element in the factor space $X^{\otimes i}/R_i$. Since α is an algebra homomorphism, we have

$$\begin{aligned} \alpha(\bar{x}_i\bar{x}_j) &= \alpha(\overline{x_i \otimes x_j}) \\ &= \alpha_{i+j}(\overline{x_i \otimes x_j}) + \alpha_{i+j+1,i+j}(\overline{x_i \otimes x_j}) + \cdots \\ &= \alpha(\bar{x}_i)\alpha(\bar{x}_j) \\ &= \big\{\alpha_i(\bar{x}_i) + \alpha_{i+1,i}(\bar{x}_i) + \cdots\big\}\big\{\alpha_j(\bar{x}_j) + \alpha_{j+1,j}(\bar{x}_j) + \cdots\big\} \\ &= \alpha_i(\bar{x}_i)\alpha_j(\bar{x}_j)\big\{\alpha_i(\bar{x}_i)\alpha_{j+1,j}(\bar{x}_j) + \alpha_{i+1,j}(\bar{x}_i)\alpha_j(\bar{x}_j)\big\} + \cdots \end{aligned}$$

and, hence, $\alpha_{i+j}(\overline{x_i \otimes x_j}) = \alpha_i(\bar{x}_i)\alpha_j(\bar{x}_j)$. Therefore, the diagonal part

$$\alpha' = \begin{bmatrix} 1 & & & & \mathbf{0} \\ & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \alpha_d \end{bmatrix}$$

gives a K-algebra isomorphism $\Lambda(\Phi') \xrightarrow{\sim} \Lambda(\Phi)$ also. Hence, we may suppose $\alpha = \alpha'$ from the beginning, and therefore, we obtain the commutative diagrams

$$\begin{tikzcd} X^{\otimes i} \arrow[r, "\alpha_1^{\otimes i}"] \arrow[d] & X^{\otimes i} \arrow[d] \\ X^{\otimes i}/R'_i \arrow[r, "\alpha_i"'] & X^{\otimes i}/R_i \end{tikzcd}$$

for all i, where the vertical maps are the canonical projections. In particular, we have the commutative diagram

$$\begin{tikzcd} X^{\otimes d} \arrow[r, "\alpha_1^{\otimes d}"] \arrow[d] & X^{\otimes d} \arrow[d] \\ X^{\otimes d}/R'_d \arrow[r, "\alpha_d"] \arrow[d, "\cong"'] & X^{\otimes d}/R_d \arrow[d, "\cong"] \\ K \arrow[r, "(c?)"'] & K. \end{tikzcd}$$

Let us put $s = (1/w) \cdot \alpha_1 \in GL(X)$, where $w \in K$ such that $w^d = c$. Then, composing vertical maps in the above diagram, we have the commutative diagram

$$\begin{tikzcd} X^{\otimes d} \arrow[r, "s^{\otimes d}"] \arrow[d, "\theta'"'] & X^{\otimes d} \arrow[d, "\theta"] \\ K \arrow[r, "\mathrm{id}"'] & K \end{tikzcd}$$

as required. $\square$

For any element $s \in GL(X)$, define the regular matrix S by $s(e_i) = \sum_{j=1}^{n} e_j S(j,i)$. Then, it is checked that $\hat{e}_j \circ s = \sum_{j=1}^{n} S(j,i)\hat{e}_i$. Using this, the map $\theta \circ s^{\otimes d}$ can be expressed as

$$\sum_{i_1,\dots,i_d} \left(\sum_{j_1,\dots,j_d} \Theta(j_1,\dots,j_d) S(j_1,i_1) \cdots S(j_d,i_d) \right) \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_d}.$$

Hence, denoting by $\Theta * S$ the generalized matrix corresponding to $\theta \circ s^{\otimes d}$, we obtain

$$(\Theta * S)(i_1, \ldots, i_d) = \sum_{j_1, \ldots, j_d} \Theta(j_1, \ldots, j_d) S(j_1, i_1) \cdots S(j_d, i_d).$$

We consider the K-vector space $H(\Gamma)$ of all generalized matrices Θ which satisfy the condition $(\star)$. Put

$$H = \bigcup_{\Gamma \in GL(n,K)} H(\Gamma) \subseteq D(X)^{\otimes d}.$$

Then, by Lemma 4.1, the group $GL(n, K)$ acts on H by

$$H(\Gamma) \ni \Theta \quad \overset{S}{\longmapsto} \quad \Theta * S \in H\bigl(S^{-1} \Gamma S\bigr).$$

Each point $\Theta \in H(\Gamma)$ determines a graded Frobenius local algebra

$$\Lambda(\Gamma, \Theta) = \Lambda(1_K, \gamma, \theta) = \Lambda(\gamma, \theta).$$

In order to construct all graded Frobenius algebras, we only need to use nondegenerate admissible systems by Proposition 2.6. We call a point $\Theta \in H$ nondegenerate if the admissible system $(1_K, \gamma, \theta)$ is nondegenerate. A point $\Theta \in H$ is nondegenerate if and only if it satisfies the condition $(\star\star)$. For a nondegenerate point $\Theta \in H$, by Lemma 4.2, we may write simply $\Lambda(\Theta) = \Lambda(\Gamma, \Theta) = \Lambda(\theta) = \Lambda(\gamma, \theta)$. By Theorem 4.3, for nondegenerate points $\Theta, \Theta' \in H$, the algebras $\Lambda(\Theta)$ and $\Lambda(\Theta')$ are isomorphic over K if and only if their $GL(n, K)$-orbits coincide. Therefore, to classify algebras $\Lambda(\Theta)$, we have to classify $GL(n, K)$-orbits of nondegenerate points Θ in H.

5. Some constructions of admissible systems

Suppose $\dim_K X = d$ for a K-space X. Then, the map $\det : X^{\otimes d} \to K$ is defined by

$$\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{d1} \end{bmatrix} \otimes \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{d2} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} x_{1d} \\ x_{2d} \\ \vdots \\ x_{dd} \end{bmatrix} \mapsto \det \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dd} \end{bmatrix},$$

where ${}^t(x_{1i}, x_{2i}, \ldots, x_{di}) = \sum_{j=1}^{d} e_j x_{ji} \in X$. It is not hard to see that the triple $(1_K, (-1)^{d-1}, \det)$ is an admissible system and the algebra $\Lambda(\det)$ is nothing but the exterior algebra $\bigwedge X$. We can generalize this construction of admissible system.

Let us denote by $\mathfrak{S}_d$ the symmetric group of d letters. We put

$$\kappa = \begin{pmatrix} 1 & 2 & \cdots & d \\ 2 & \cdots & d & 1 \end{pmatrix} \in \mathfrak{S}_d.$$

Using any subgroup G of $\mathfrak{S}_d$ containing the element κ and a group homomorphism $\rho : G \to K^\times$, we define a linear map $\theta_\rho : X^{\otimes d} \to K$ by

$$\theta_\rho(e_{i_1} \otimes \cdots \otimes e_{i_d}) = \begin{cases} \rho(\sigma) & \cdots \quad \begin{pmatrix} 1 & \cdots & d \\ i_1 & \cdots & i_d \end{pmatrix} = \sigma \in G, \\ 0 & \cdots \quad \text{otherwise.} \end{cases}$$

Further, we put

$$\gamma_\rho = 1/\rho(\kappa) = \begin{bmatrix} 1/\rho(\kappa) & & \\ & \ddots & \\ & & 1/\rho(\kappa) \end{bmatrix}.$$

Theorem 5.1. *The triple* $(1_K, \gamma_\rho, \theta_\rho)$ *is a nondegenerate admissible system.*

Proof. Write $\gamma = \gamma_\rho$ and $\theta = \theta_\rho$ simply. For an element $\sigma \in G$, we have

$$\begin{aligned} \theta\bigl(e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(d)} \otimes \gamma(e_{\sigma(1)})\bigr) &= \mathbf{1}/\rho(\kappa) \cdot \theta(e_{\sigma\circ\kappa(1)} \otimes e_{\sigma\circ\kappa(2)} \otimes e_{\sigma\circ\kappa(d)}) \\ &= \mathbf{1}/\rho(\kappa)\rho(\sigma \circ \kappa) \\ &= \rho(\sigma) \\ &= \theta(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(d)}). \end{aligned}$$

This shows that the triple $(1_K, \gamma, \theta)$ is an admissible system. Next, suppose that $\theta((\sum_{i=1}^d x_i e_i) \otimes X^{\otimes(d-1)}) = 0$. Then, since $\kappa^{t-1}(\mathbf{1}) = t$ for each $1 \leqslant t \leqslant d$, we obtain $x_t = 0$ for all t by

$$0 = \theta\left(\sum_{i=1}^d x_i e_i \otimes e_{\kappa^{t-1}(2)} \otimes \cdots \otimes e_{\kappa^{t-1}(d)}\right) = x_t \rho\bigl(\kappa^{t-1}\bigr).$$

Therefore, the map $\Theta = \theta_\rho$ is nondegenerate. □

It is easy to see that the linear map det coincides with the map θ_{sgn} for the group homomorphism $\mathrm{sgn} : G = \mathfrak{S}_d \to \{\pm 1\} \subseteq K^\times$. There is another generalization of exterior algebras. Denote by $K\langle T_1, \ldots, T_d\rangle$ the noncommutative associative K-algebra freely generated by d invariables $T_1, \ldots, T_d$. It is known and easily seen that the factor algebra

$$\Lambda_{\{q_{ij}\}} = K\langle T_1, \ldots, T_d\rangle \big/ \bigl(T_i^2, q_{ij} \cdot T_i T_j - T_j T_i \mid i < j\bigr)$$

is a graded Frobenius algebra for any set of scalars $\{0 \neq q_{ij} \in K \mid i < j\}$. We can describe this algebra by using a linear map.

Set

$$\lambda_i = \prod_{j<i} q_{ji} \Big/ \prod_{i<j} q_{ij}$$

for all $1 \leqslant i \leqslant d$, where we understand that $\prod_{i<j} q_{ij} = 1$ if there does not exist a pair $i < j$. Define the automorphism $\gamma_{\{q_{ij}\}} \in GL(X)$ by

$$\gamma_{\{q_{ij}\}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}.$$

We consider a linear map $\theta_{\{q_{ij}\}} : X^{\otimes d} \to K$ defined by

$$\theta_{\{q_{ij}\}}(e_{i_1} \otimes \cdots \otimes e_{i_d}) = \begin{cases} \prod_{s<t,\ i_s>i_t} q_{i_t i_s} & \ldots\ \{i_1, \ldots, i_d\} = \{1, \ldots, d\}, \\ 0 & \ldots\ \text{otherwise}, \end{cases}$$

where we understand $\prod_{s<t,\ i_s>i_t} q_{i_t i_s} = 1$ if there does not exist a pair $s < t$ with $i_s > i_t$.

Proposition 5.2. *The triple* $(1_K, \gamma_{\{q_{ij}\}}, \theta_{\{q_{ij}\}})$ *is a nondegenerate admissible system.*

Proof. Write simply as $\gamma = \gamma_{\{q_{ij}\}}$ and $\theta = \theta_{\{q_{ij}\}}$. In order to prove that the triple $(1_K, \gamma, \theta)$ is an admissible system, we have to check the equality

$$\theta(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}) = \lambda_{i_1} \cdot \theta(e_{i_2} \otimes \cdots \otimes e_{i_d} \otimes e_{i_1})$$

for any $\{i_1, i_2, \ldots, i_d\} = \{1, 2, \ldots, d\}$. This is obvious, because we have

$$\theta(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}) = \begin{cases} 1/q_{i_1 i_2} \cdot \theta(e_{i_2} \otimes e_{i_1} \otimes \cdots \otimes e_{i_d}) & \ldots\ i_1 < i_2, \\ q_{i_2 i_1} \cdot \theta(e_{i_2} \otimes e_{i_1} \otimes \cdots \otimes e_{i_d}) & \ldots\ i_1 > i_2, \end{cases}$$

by the definition of the map θ, and this process can be repeated until we get $\lambda_{i_1} \cdot \theta(e_{i_2} \otimes \cdots \otimes e_{i_d} \otimes e_{i_1})$. Next, suppose that $\theta((\sum_{i=1}^d x_i e_i) \otimes X^{\otimes(d-1)}) = 0$. Then, we have

$$\begin{aligned} 0 &= \theta\left(\left(\sum_{i=1}^d x_i e_i\right) \otimes e_1 \otimes \cdots \otimes \check{e}_t \otimes \cdots \otimes e_d\right) \\ &= x_t \cdot \theta(e_t \otimes e_1 \otimes \cdots \otimes \check{e}_t \otimes \cdots \otimes e_d) = x_t \cdot \prod_{0<i<t} q_{it} \end{aligned}$$

and $x_t = 0$ for all $1 \leqslant t \leqslant d$, where $\check{e}_t$ means that e_t is missing. Therefore, we get $\sum_{i=1}^d x_i e_i = 0$. Hence, the map θ is nondegenerate. □

It is not hard to see that the K-algebra $\Lambda_{\{q_{ij}\}}$ is isomorphic to $\Lambda(\theta_{\{q_{ij}\}})$. In general, the automorphism $\gamma_{\{q_{ij}\}}$ is not a scalar, hence, by Theorem 4.3, the algebra $\Lambda(\theta_{\{q_{ij}\}})$ is different from the algebra $\Lambda(\theta_\rho)$ defined before. There is a way to construct many admissible systems by twisting a given system. It is easy to prove the following assertion.

Proposition 5.3. *Let* $(1_K, \gamma, \theta)$ *be a nondegenerate admissible system. Suppose that an element* $s \in GL(X)$ *satisfies* $\theta = \theta \circ s^{\otimes d}$*. Then, the triple*

$$\left(1_K, s^{-d} \circ \gamma, \theta \circ \left(1 \otimes s \otimes \cdots \otimes s^{\otimes d-1}\right)\right)$$

is a nondegenerate admissible system.

Consider the subgroup $G(\theta) = \{s \in GL(X) \mid \theta = \theta \circ s^{\otimes d}\} \leqslant GL(X)$. It is immediate by Lemmas 4.2 that $G(\theta) \leqslant C(\gamma) = \{s \in GL(X) \mid \gamma = s^{-1} \circ \gamma \circ s\}$. We denote by θ^s the linear map $\theta \circ (1 \otimes s \otimes \cdots \otimes s^{d-1})$ for $s \in G(\theta)$. It is obvious that $G(\det) = SL(X)$.

Proposition 5.4. *The* K*-algebras* $\Lambda(\det^s)$ *and* $\Lambda(\det^t)$ *are isomorphic if and only if* $t = w \cdot g^{-1} \circ s \circ g$ *for some* $g \in GL(X)$ *and* $w \in K$ *such that* $w^d = 1$.

Proof. Suppose $\det^t = \det^s \circ g^{\otimes d}$ for $g \in GL(X)$. Then, we see that $\det[x_1, t(x_2), \ldots, t^{d-1}(x_d)]$ coincides with

$$\det G \cdot \det\left[x_1, \left(g^{-1} \circ s \circ g\right)(x_2), \left(g^{-1} \circ s \circ g\right)^{d-1}(x_d)\right]$$

for $x_i = {}^t(x_{1i}, x_{2i}, \ldots, x_{di})$, where G is the regular matrix corresponding to g. In this equation, if we put $x_1 = (g^{-1} \circ s \circ g)(x_2)$, then we get

$$\det\left[\left(g^{-1} \circ s \circ g\right)(x_2), t(x_2), t^2(x_3), \ldots, t^{d-1}(x_d)\right] = 0$$

for arbitrary column vectors $x_2, x_3, \ldots, x_d$. From here, we have $t = w \cdot g^{-1} \circ s \circ g$ for some $w \in K$, and $w^d = 1$ holds because both t and $g^{-1} \circ s \circ g$ are members of $SL(X)$. □

We next consider the ideals $R(\theta)$ and $R(\theta^s)$ for a nondegenerate admissible system $(1_K, \gamma, \theta)$ and $s \in G(\theta)$.

Proposition 5.5. *Let* $(1_K, \gamma, \theta)$ *be a nondegenerate admissible system with the map* $\theta : X^{\otimes d} \to K$*. Suppose* $\dim_K X \geqslant 2$*. Then, we have*

$$X \otimes R_d(\theta) + R_d(\theta) \otimes X = X^{\otimes(d+1)}.$$

Proof. Tensoring X to the exact sequence

$$0 \to R_d(\theta) \to X^{\otimes d} \xrightarrow{\theta} K \to 0,$$

we have the following two exact sequences:

$$0 \to X \otimes R_d(\theta) \to X^{\otimes(d+1)} \xrightarrow{X \otimes \theta} X \to 0$$

and

$$0 \to R_d(\theta) \otimes X \to X^{\otimes(d+1)} \xrightarrow{\theta \otimes X} X \to 0.$$

Then, the condition $X \otimes R_d(\theta) + R_d(\theta) \otimes X = X^{\otimes(d+1)}$ is equivalent to saying that the push-out of the linear maps $X \otimes \theta$ and $\theta \otimes X$ is zero. In order to prove this, for a pair of linear maps $f, g \in \mathrm{Hom}_K(X, K)$ satisfying $f \circ X \otimes \theta = g \circ \theta \otimes X$, it is enough to show $f = 0 = g$. For any element $x \otimes z \otimes y \in X^{\otimes(d+1)}$ with $x, y \in X$ and $z \in X^{\otimes(d-1)}$, we have

$$f(x)\theta(z \otimes y) = f \circ X \otimes \theta(x \otimes z \otimes y) = g \circ \theta \otimes X(x \otimes z \otimes y) = \theta(x \otimes z)g(y).$$

Therefore, if $f(x) = 0$ then $\theta(x \otimes z)g(y) = 0$ holds for any $z \in X^{\otimes(d-1)}$ and $y \in X$. Suppose $g \neq 0$ and $g(y) = 0$. Then, $\theta(x \otimes z) = 0$ for any $z \in X^{\otimes(d-1)}$, and hence, $x = 0$ since θ is nondegenerate. This contradicts $\dim_K X \geqslant 2$. Hence, $g = 0$ and then $f = 0$ follows, since $X \otimes \theta$ is surjective. □

For an element $s \in G(\theta)$ and $0 \leqslant i \leqslant d$, we have the commutative diagrams

$$\begin{tikzcd} X^{\otimes i} \arrow[r, "\theta_i"] \arrow[d, "s^{\otimes i}"'] & D\left(X^{\otimes(d-i)}\right) \\ X^{\otimes i} \arrow[r, "\theta_i"'] & D\left(X^{\otimes(d-i)}\right) \arrow[u, "D(s^{\otimes(d-i)})"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X^{\otimes i} \arrow[r, "\theta_i^s"] \arrow[d, "1\otimes\cdots\otimes s^{i-1}"'] & D\left(X^{\otimes(d-i)}\right) \\ X^{\otimes i} \arrow[r, "\theta_i"'] & D\left(X^{\otimes(d-i)}\right), \arrow[u, "D(s^i\otimes\cdots\otimes s^{d-i})"'] \end{tikzcd}$$

as easily checked. Hence, for each i, we obtain two bijections $s^{\otimes i} : R_i(\theta) \to R_i(\theta)$ and $1 \otimes \cdots \otimes s^{i-1} : R_i(\theta^s) \to R_i(\theta)$.

Proposition 5.6. *For an element $s \in G(\theta)$ and for each i, the condition*

$$R_i(\theta) = X \otimes R_{i-1}(\theta) + R_{i-1}(\theta) \otimes X$$

implies

$$R_i\left(\theta^s\right) = X \otimes R_{i-1}\left(\theta^s\right) + R_{i-1}\left(\theta^s\right) \otimes X.$$

Proof. We have

$$\begin{aligned} \left(1 \otimes s \otimes \cdots \otimes s^{i-1}\right)\left(X \otimes R_{i-1}\left(\theta^s\right)\right) &= X \otimes \left(s \otimes \cdots \otimes s^{i-2} \otimes s^{i-1}\right)\left(R_{i-1}\left(\theta^s\right)\right) \\ &= X \otimes \left(1 \otimes \cdots \otimes s^{i-2}\right)\left(s^{i-1}\left(R_{i-1}\left(\theta^s\right)\right)\right) \\ &= X \otimes \left(1 \otimes \cdots \otimes s^{i-2}\right)\left(R_{i-1}\left(\theta^s\right)\right) \\ &= X \otimes R_{i-1}(\theta). \end{aligned}$$

Similarly, we obtain

$$\left(1 \otimes \cdots \otimes s^{i-1}\right)\left(R_{i-1}\left(\theta^s\right) \otimes X\right) = R_{i-1}(\theta) \otimes X.$$

Therefore, we get $R_i(\theta^s) = X \otimes R_{i-1}(\theta^s) + R_{i-1}(\theta^s) \otimes X$, since the restricted map $1 \otimes \cdots \otimes s^{i-1} : R_i(\theta^s) \to R_i(\theta)$ is bijective. $\square$

Corollary 5.7. *If the algebra $\Lambda(\theta)$ is quadratic then so is $\Lambda(\theta^s)$.*

Finally we consider whether the twisted algebra is Koszul. For Koszul algebras, we refer to Beilinson, Ginsburg, and Soergel [2].

Theorem 5.8. *Let $(1_K, \gamma, \theta)$ be a nondegenerate admissible system such that the algebra $\lambda(\theta)$ is Koszul. Then, for any element $s \in G(\theta)$, the algebra $\Lambda(\theta^s)$ is Koszul again.*

Proof. By our assumption, $D(\Lambda^!(\theta)) \otimes \Lambda(\theta)$ gives a graded projective resolution of the trivial module $K_{\Lambda(\theta)}$, where we consider $D(\Lambda^!(\theta)) \otimes \Lambda(\theta)$ as a natural bigraded space and D for the infinite-dimensional space $\Lambda^!(\theta)$ means the K-dual of a graded space. For each $i, j \geqslant 0$, we have a bijection

$$\mu_{j,i} = D\left(\bar{s}^{d-i+1} \otimes \cdots \otimes \bar{s}^{d-i+j}\right) \otimes \left(\bar{s}^{d-i+1} \otimes \cdots \otimes \bar{s}^{d}\right)$$

from $D(\Lambda^!(\theta^s)) \otimes \Lambda^!_j(\theta^s)$ to $D(\Lambda^!_j(\theta)) \otimes \Lambda_i(\theta)$, where

$$\Lambda^!_j\left(\theta^s\right) = D(X)^{\otimes j} \Big/ \sum_{\alpha+\beta=j-2} D(X)^{\otimes\alpha} \otimes R_2^{\perp}\left(\theta^s\right) \otimes D(X)^{\otimes\beta},$$

$$\Lambda_i\left(\theta^s\right) = X^{\otimes i} \Big/ \sum_{\alpha+\beta=i-2} X^{\otimes\alpha} \otimes R_2\left(\theta^s\right) \otimes X^{\otimes\beta}$$

and

$$\Lambda^!_j(\theta) = D(X)^{\otimes j} \Big/ \sum_{\alpha+\beta=j-2} D(X)^{\otimes\alpha} \otimes R_2^{\perp}(\theta) \otimes D(X)^{\otimes\beta},$$

$$\Lambda_i(\theta) = X^{\otimes i} \Big/ \sum_{\alpha+\beta=i-2} X^{\otimes\alpha} \otimes R_2(\theta) \otimes X^{\otimes\beta}.$$

It is a routine work to check the commutativity of the diagrams

$$\begin{tikzcd}
D\left(\Lambda^!_j\left(\theta^s\right)\right) \otimes \Lambda_i\left(\theta^s\right) \arrow[r] \arrow[d, "\mu_{j+1,i}"'] & D\left(\Lambda^!_j\left(\theta^s\right)\right) \otimes \Lambda_{i+1}\left(\theta^s\right) \arrow[d, "\mu_{j,i-1}"'] \\
D\left(\Lambda^!_j(\theta)\right) \otimes \Lambda_i(\theta) \arrow[r] & D\left(\Lambda^!_j(\theta)\right) \otimes \Lambda_{i+1}(\theta).
\end{tikzcd}$$

Therefore, $D(\Lambda^!(\theta^s)) \otimes \Lambda(\theta^s)$ also gives a graded projective resolution of $K_{\Lambda(\theta^s)}$ and the algebra $\Lambda(\theta^s)$ is Koszul. $\square$

References

[1] M. Artin, W. Schelter, Graded algebras of global dimension 3, Adv. Math. 66 (1987) 171–216.
[2] A. Beilinson, V. Ginsburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996) 473–527.
[3] R. Martinéz-Villa, Graded, self-injective and Koszul algebras, Preprint, 1997.
[4] S. Smith, Some finite dimensional algebras related to elliptic curves, in: Representation Theory of Algebras and Related Topics, CMS Conf. Proc., Vol. 19, American Mathematical Society, Providence, RI, 1996, pp. 315–348.
[5] T. Wakamatsu, Tilting theory and self-injective algebras, in: Finite Dimensional Algebras and Related Topics, Kluwer Academic, 1994, pp. 361–390.